# Exercise 10

Convert each of the following Fredholm integral equation in 9–16 to an equivalent BVP:

$$u(x) = 3x^2 + \int_0^1 K(x,t)u(t)\,dt,\ K(x,t) = \begin{cases} t(1-x), & \text{for } 0 \le t \le x \\ x(1-t), & \text{for } x \le t \le 1 \end{cases}$$

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### Solution

Substitute the given kernel $K(x,t)$ into the integral.

$$u(x) = 3x^2 + \int_0^x t(1-x)u(t)\,dt + \int_x^1 x(1-t)u(t)\,dt \tag{1}$$

Differentiate both sides with respect to $x$.

$$u'(x) = 6x + \frac{d}{dx}\int_0^x t(1-x)u(t)\,dt + \frac{d}{dx}\int_x^1 x(1-t)u(t)\,dt$$

Apply the Leibnitz rule to differentiate the integrals.

$$\begin{aligned} &= 6x + \int_0^x \frac{\partial}{\partial x} t(1-x)u(t)\,dt + \cancel{x(1-x)u(x)\cdot 1} - (0)(1-x)u(0)\cdot 0 \\ &\qquad + \int_x^1 \frac{\partial}{\partial x} x(1-t)u(t)\,dt + x(0)u(1)\cdot 0 - \cancel{x(1-x)u(x)\cdot 1} \\ &= 6x + \int_0^x (-t)u(t)\,dt + \int_x^1 (1-t)u(t)\,dt \\ &= 6x - \int_0^x tu(t)\,dt - \int_1^x (1-t)u(t)\,dt \end{aligned}$$

Differentiate both sides with respect to $x$ once more.

$$\begin{aligned} u''(x) &= 6 - \frac{d}{dx}\int_0^x tu(t)\,dt - \frac{d}{dx}\int_1^x (1-t)u(t)\,dt \\ &= 6 - xu(x) - (1-x)u(x) \\ &= 6 - \cancel{xu(x)} - u(x) + \cancel{xu(x)} \end{aligned}$$

The boundary conditions are found by setting $x = 0$ and $x = 1$ in equation (1).

$$\begin{aligned} u(0) &= 3(0)^2 + \int_0^0 t(1)u(t)\,dt + \int_0^1 (0)(1-t)u(t)\,dt = 0 \\ u(1) &= 3(1)^2 + \int_0^1 t(0)u(t)\,dt + \int_1^1 (1)(1-t)u(t)\,dt = 3 \end{aligned}$$

Therefore, the equivalent BVP is

$$u'' + u = 6,\ u(0) = 0,\ u(1) = 3.$$